Binomial theorem Vs Trinomial Theorem

Yue Kwok Choy

The coefficients of x^2 and x^3 in the trinomial expansion of $(1 + kx + 2x^2)^{10}$ are 425 and 3780 respectively. Find the value of k and the coefficients of x^4.

Method 1 (Use Binomial Theorem twice)

$$(1 + kx + 2x^2)^{10} = [1 + (kx + 2x^2)]^{10}$$
$$= 1 + 10(kx + 2x^2) + \frac{10 \times 9}{2}(kx + 2x^2)^2 + \frac{10 \times 9 \times 8}{3 \times 2 \times 1}(kx + 2x^2)^3 + \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}(kx + 2x^2)^4 + \cdots$$
$$= 1 + 10(kx + 2x^2) + 45[kx^2 + 2(kx)(2x^2) + 4x^4] + 120[(kx)^3 + 3(kx)^2(2x^2) + \cdots] + 210[(kx)^4 + \cdots] + \cdots$$
$$= 1 + 10(kx + 2x^2) + 45(k^2 x^2 + 4\,k\,x^3 + 4\,x^4) + 120(k^3 x^3 + 6k^2 x^4 + \cdots) + 210(k^4 x^4 + ..) + \cdots$$
$$= 1 + (10\,k\,x + 20\,x^2) + (45\,k^2x^2 + 180\,kx^3 + 180\,x^4) + (120\,k^3\,x^3 + 720\,k^2\,x^4 + ..) + (210\,k^4\,x^4 + ..) + \cdots$$

Coefficient of $x^2 = 20 + 45\,k^2 = 425$
Coefficient of $x^3 = 180\,k + 120\,k^3 = 3780$

Solving, we get $k = 3$
The coefficients of $x^4 = 180 + 720\,k^2 + 210\,k^4 = 180 + 720\,(3^2) + 210\,(3^4) = 23670$

Method 2 (Use Binomial Theorem twice)

$$(1 + kx + 2x^2)^{10} = [(1 + kx) + 2x^2]^{10}$$
$$= (1 + kx)^{10} + 10(1 + kx)^9(2x^2) + 45(1 + kx)^8(2x^2)^2 + \cdots$$
$$= (1 + 10\,k\,x + 45\,k^2\,x^2 + 120\,k^3\,x^3 + 210\,k^4\,x^4 + \cdots) + 10(1 + 9\,k\,x + 36\,k^2\,x^2 + 84\,k^3\,x^3 + \cdots)(2x^2) + (1 + \cdots)(4x^4) + \cdots$$

Coefficient of $x^2 = 20 + 45\,p^2 = 425$
Coefficient of $x^3 = 180\,k + 120\,k^3 = 3780$

Solving, we get $k = 3$
The coefficients of $x^4 = 180 + 720\,k^2 + 210\,k^4 = 180 + 720\,(3^2) + 210\,(3^4) = 23670$

Method 3 (Trinomial method)

Term of $x^2 = \binom{10}{8,2,0}(1)^8(kx)^2(2x^2)^0 + \binom{10}{9,0,1}(1)^9(kx)^0(2x^2)^1 = 425x^2$

$\frac{10!}{8!2!0!}k^2 + \frac{10!}{9!0!1!}(2) = 425 \Longrightarrow 20 + 45\,k^2 = 425$

Term of $x^3 = \binom{10}{7,3,0}(1)^7(kx)^3(2x^2)^0 + \binom{10}{8,1,1}(1)^8(kx)^1(2x^2)^1 = 3780x^3$

$\frac{10!}{7!3!0!}k^3 + \frac{10!}{8!1!1!}(2k) = 3780 \Longrightarrow 180\,k + 120\,k^3 = 3780$

Solving, we get $k = 3$,

$(1 + 3x + 2x^2)^{10} = (1 + x)^{10}(1 + 2x)^{10}$
$= (1 + 10x + 45\,x^2 + 120\,x^3 + 210\,x^4 + ..)(1 + 20\,x + 180\,x^2 + 960\,x^3 + 3360\,x^4 + ..)$

The coefficients of x^4
$=(1)(3360) + (10)(960) + (45)(180) + (120)(20) + (210)(1) = 23670$

(Alternatively)

The term of x^4
$= \frac{10!}{8!0!2!}(1)^8(3x)^0(2x^2)^2 + \frac{10!}{7!2!1!}(1)^7(3x)^2(2x^2)^1 + \frac{10!}{6!4!0!}(1)^6(3x)^4(2x^2)^0 = 23670\,x^4$

The coefficients of $x^4 = 23670$